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Fuzzy torus via q -Parafermion

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Abstract

We note that the recently introduced fuzzy torus can be regarded as a q -deformed parafermion. Based on this picture, classification of the Hermitian representations of the fuzzy torus is carried out. The result involves Fock-type representations and new finite-dimensional representations for q being a root of unity as well as already known finite-dimensional ones.

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1. Introduction

Fuzzy spaces have been widely accepted as models of noncommutative manifolds in the context of quantum field theory (see for example [1]). The fuzzy 2-sphere [2] has been studied extensively. Regarding the ordinary 2-sphere S^2 as a one-dimensional complex projective space $\mathbb{C}P^1$, construction of the fuzzy S^2 has been extended to higher dimensional fuzzy projective spaces [3]. Many of the fuzzy spaces obtained so far are related to the complex projective space $\mathbb{C}P^n$. Supersymmetric extensions of the fuzzy S^2 [4] and the fuzzy $\mathbb{C}P^n$ [5] embody such relations. Exploiting the observation that $\mathbb{C}P^3$ is an S^2 bundle over S^4 , the fuzzy S^4 has been constructed [6, 7]. Fuzzy S^5 has been established by utilizing a $U(1)$ fibration over a fuzzy $\mathbb{C}P^2$ [8]. Further examples are found in the investigations of fuzzy unitary Grassmannian spaces [9, 10] and fuzzy orthogonal Grassmannian spaces [11]. Furthermore, fuzzy versions of various toric varieties have been considered in [12] by embedding such varieties in $\mathbb{C}P^n$.

The sphere S^2 is a Riemann surface of genus zero that may be embedded in \mathbb{R}^3 . Using the Cartesian coordinates (x, y, z) in \mathbb{R}^3 it is described by a constrained polynomial $C(x, y, z) = 0$, where $C(x, y, z) = x^2 + y^2 + z^2 - 1$. Similar polynomial description of compact connected Riemann surfaces of arbitrary genus has been introduced in [13]. These authors have developed a general recipe for constructing fuzzy Riemann surfaces. Explicit realization for the torus (genus one) is completed by defining the fuzzy torus as a nonlinear C -algebra with three generators [13]. The fuzzy torus algebra T_F^2 has been recast as a q -deformed

Lie algebra in [14]. This may be understood as a linearization of the nonlinear T_F^2 algebra by using the q -deformed commutators.

In the present work, we are looking at the fuzzy torus algebra from a different viewpoint. We regard T_F^2 as a q -deformed parafermion algebra which is different from the q -deformations of parafermion or paraboson discussed so far [15–20]. Using this approach, we classify the Hermitian representations of T_F^2 . In addition to recovering the known finite-dimensional representations [13, 14], some new and nontrivial representations are found. A Fock-type infinite-dimensional representation and a finite-dimensional representation, for q being a root of unity, are produced. The classification of the Hermitian representations will facilitate deeper understanding of T_F^2 and model building of quantum field theory on the fuzzy torus. In the following section, we recall the definition of T_F^2 and explain the interpretation as q -deformed parafermion. Representations of T_F^2 are systematically constructed under simple assumptions in section 3 so as to achieve the classification. Our concluding remarks are given in section 4.

2. Fuzzy torus and deformed parafermion

The polynomial description [13] of the torus comprising three real commuting variables reads

$$C(x, y, z) = (x^2 + y^2 - \mu)^2 + z^2 - c, \quad (2.1)$$

where μ and $c(>0)$ are real parameters. In [13], the parameter c is set equal to unity. Following [14], we, however, keep its value arbitrary. The relation $C(x, y, z) = 0$ describes a surface in \mathbb{R}^3 , which is a squashed sphere for the domain $-\sqrt{c} < \mu < \sqrt{c}$, and a torus for $\sqrt{c} < \mu$.

Fuzzy analogue of the surface is introduced by replacing the commuting variables (x, y, z) with the generating elements (X, Y, Z) that satisfy the commutation relations [13]

$$[X, Y] = i\hbar Z, \quad [Y, Z] = i\hbar\{X, \varphi\}, \quad [Z, X] = i\hbar\{Y, \varphi\}, \quad (2.2)$$

$$\varphi \equiv X^2 + Y^2 - \mu, \quad (2.3)$$

and the constraint

$$C_F \equiv \varphi^2 + Z^2 = c. \quad (2.4)$$

The parameter \hbar imparts noncommutativity. Its commutative limit is given by $\hbar \rightarrow 0$. The associative algebra T_F^2 , generated by the variables (X, Y, Z) obeying relations (2.2), (2.3), satisfies the Jacobi identity. The element C_F is the center of the algebra.

Introducing the complex variable $W = X + iY$ and its Hermitian adjoint W^\dagger , the defining relations (2.2), (2.3) are reexpressed as

$$[W, W^\dagger] = 2\hbar Z, \quad [Z, W] = \hbar\{W, \varphi\}, \quad [Z, W^\dagger] = -\hbar\{W^\dagger, \varphi\}, \quad (2.5)$$

$$\varphi \equiv \frac{1}{2}\{W, W^\dagger\} - \mu. \quad (2.6)$$

Eliminating Z via the first equation in (2.5), it is observed [13] that the operators W, W^\dagger obey the following trilinear relation and its Hermitian adjoint:

$$W(W^\dagger)^2 - 2\frac{1 - \hbar^2}{1 + \hbar^2}W^\dagger W W^\dagger + (W^\dagger)^2 W = \mu\frac{4\hbar^2}{1 + \hbar^2}W^\dagger. \quad (2.7)$$

The deformation parameter q is introduced [14] by the relation

$$2\frac{1 - \hbar^2}{1 + \hbar^2} = q + q^{-1}. \quad (2.8)$$

It follows that the complex parameter q is of unit magnitude: $|q| = 1$. The commutative $\hbar \rightarrow 0$ limit now corresponds to $q \rightarrow 1$. We now scale the conjugate variables W and W^\dagger as follows:

$$a = \left(\frac{2}{|\mu|(2-q-q^{-1})} \right)^{1/2} W, \quad a^\dagger = \left(\frac{2}{|\mu|(2-q-q^{-1})} \right)^{1/2} W^\dagger. \quad (2.9)$$

For a fixed μ above scaling is well defined in the *noncommutative* $q \neq 1$, i.e. $\hbar \neq 0$ regime. The trilinear relation (2.7) now assumes two distinct forms depending on the sign of the parameter μ . For the choice $\mu < 0$, it reads

$$a(a^\dagger)^2 - [2]_q a^\dagger a a^\dagger + (a^\dagger)^2 a = -2a^\dagger, \quad (2.10)$$

where the q -number is defined as usual

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (2.11)$$

For the alternate $\mu > 0$ case, relation (2.7) may be recast as

$$a(a^\dagger)^2 - [2]_q a^\dagger a a^\dagger + (a^\dagger)^2 a = 2a^\dagger. \quad (2.12)$$

The trilinear relation (2.10) may be regarded as q -deformation of the parafermion [21]. Relation (2.12) has the same form albeit with an opposite sign on the right-hand side. One may regard this as a variant of the q -deformed parafermion. It becomes evident in the following section that the q -parafermionic picture observed in (2.10) and (2.12) facilitates a systematic study of the representations. The transformation (2.9) is singular for a fixed μ in the limit $q \rightarrow 1$. In this scenario the classical $q = 1$ parafermions do not arise. There is, however, an interesting possibility

$$|\mu| \sim (2 - q - q^{-1})^{-1}, \quad (2.13)$$

which allows taking the $q \rightarrow 1$ limit in (2.9). This limit is pertinent in understanding some of the representations discussed below.

3. Hermitian representations

In this section, we obtain all Hermitian representations of the algebra defined by (2.2). Our strategy is as follows: we start with the smallest number of assumptions and construct the most general representation. Then, further assumptions are imposed to obtain possible subclasses of the representations. As observed in [13], the operators aa^\dagger and $a^\dagger a$ commute, and, therefore, they are simultaneously diagonalizable. We thus assume the existence of a normalized state $|0\rangle$ such that

$$aa^\dagger|0\rangle = p|0\rangle, \quad a^\dagger a|0\rangle = r|0\rangle, \quad \langle 0|0\rangle = 1. \quad (3.1)$$

Other states may be obtained by repeated actions of a or a^\dagger on $|0\rangle$. The states are required to be normalizable:

$$|n\rangle = \mathcal{N}_n (a^\dagger)^n |0\rangle, \quad |-n\rangle = \mathcal{N}_{-n} a^n |0\rangle \quad \forall n > 0, \quad (3.2)$$

where $\mathcal{N}_{\pm n}$ are the normalization constants. Normalization of the $|\pm 1\rangle$ states puts constraints on the values of p and r :

$$\| |1\rangle \|^2 = |\mathcal{N}_1|^2 \langle 0|aa^\dagger|0\rangle = |\mathcal{N}_1|^2 p, \quad \| |-1\rangle \|^2 = |\mathcal{N}_{-1}|^2 \langle 0|a^\dagger a|0\rangle = |\mathcal{N}_{-1}|^2 r.$$

It follows that $p, r > 0$, and we choose $\mathcal{N}_1 = p^{-1/2}$, $\mathcal{N}_{-1} = r^{-1/2}$. For the states of $|\pm n\rangle \forall n \geq 2$, we use relations (2.10) or (2.12) appropriately for evaluating the norm. For

the regime $\mu < 0$, the following relations are verified inductively:

$$a|n\rangle = \left([n]_q p - [n-1]_q r - 2 \left[\frac{n-1}{2} \right]_q [n]_{\sqrt{q}} \right)^{1/2} |n-1\rangle \equiv A_n |n-1\rangle, \quad (3.3)$$

$$a^\dagger|n\rangle = A_{n+1}|n+1\rangle, \quad (3.4)$$

and

$$a|-n\rangle = A_{-n}|-n-1\rangle, \quad a^\dagger|-n\rangle = A_{-n+1}|-n+1\rangle. \quad (3.5)$$

The diagonal operators read

$$a^\dagger a|\pm n\rangle = A_{\pm n}^2|\pm n\rangle, \quad a a^\dagger|\pm n\rangle = A_{\pm n+1}^2|\pm n\rangle. \quad (3.6)$$

For the choice $\mu > 0$, the construction of states proceeds as before:

$$a|n\rangle = \left([n]_q p - [n-1]_q r + 2 \left[\frac{n-1}{2} \right]_q [n]_{\sqrt{q}} \right)^{1/2} |n-1\rangle \equiv A_n |n-1\rangle, \quad (3.7)$$

$$a^\dagger|n\rangle = A_{n+1}|n+1\rangle, \quad (3.8)$$

while relations (3.5) and (3.6) hold with the replacement of the normalization constant A_n by A_n . The Casimir operator is quartic in the variables W, W^\dagger . Its eigenvalues may be computed via the results obtained above:

$$\frac{4}{\mu^2(2 - [2]_q)} C_F|\pm n\rangle = \begin{cases} \left(p^2 + r^2 - [2]_q pr + 2(p+r) + \frac{4}{2 - [2]_q} \right) |\pm n\rangle, & \mu < 0, \\ \left(p^2 + r^2 - [2]_q pr - 2(p+r) + \frac{4}{2 - [2]_q} \right) |\pm n\rangle, & \mu > 0. \end{cases} \quad (3.9)$$

We have *formally* obtained a double-sided infinite-dimensional representation with two parameters: p, r . Our representation has close kinship with the symplecton realization [22] of the boson calculus. Reflecting the symmetry of (2.10) (or (2.12)) and its adjoint under the exchange of a and a^\dagger , the representation has the symmetry structure $A_{n+1}(p, r) = A_{-n}(r, p)$. The usefulness of the above representations becomes evident below where we impose restrictions for obtaining the subclasses of the above infinite-dimensional representation. These restrictions terminate the infinite series of state vectors.

Assuming that the state $|0\rangle$ is annihilated by the operator a , we have, $r = 0$. All states $|-n\rangle$ labeled by negative integers are eliminated while the semi-infinite series of states $\{|n\rangle|n = 0, 1, 2, \dots, \infty\}$ are retained. This is a Fock-type representation constructed on the vacuum $|0\rangle$. The number of parameters is reduced to 1.

A key requirement for the existence for the classes of representations discussed below is that their *Hermiticity needs to be maintained*. This, in turn, requires the constants $A_{\pm n}^2, A_{\pm n}^2 \forall n > 0$ to be *non-negative*. Keeping this in mind, we study all the possibilities in order.

(1) Truncated Fock-type representation

To obtain finite-dimensional Fock-type representations, we further assume that there exists a positive integer N such that

$$A_N = 0 \quad \text{for } \mu < 0, \quad A_N = 0 \quad \text{for } \mu > 0. \quad (3.10)$$

The representation space is spanned by N -independent states: $|0\rangle, |1\rangle, \dots, |N-1\rangle$ with the highest state satisfying $a^\dagger|N-1\rangle = 0$. For instance, the N -dimensional representation for $\mu < 0$ reads

$$a^\dagger = \begin{pmatrix} 0 & A_{N-1} & 0 & \cdots & 0 \\ & 0 & A_{N-2} & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & 0 & A_1 \\ & & & & 0 \end{pmatrix}. \tag{3.11}$$

The null condition (3.10) requires

$$[N]_q p \mp 2 \left[\frac{N-1}{2} \right]_q [N]_{\sqrt{q}} = 0, \tag{3.12}$$

where the upper (lower) sign refers to $\mu < 0$ ($\mu > 0$). Relation (3.12) holds for two cases.

(i) A generic value of

$$q = \exp(i\theta) \tag{3.13}$$

leads to an N -dependent-order parameter p :

$$p = \pm 2 \left[\frac{N-1}{2} \right]_q \frac{[N]_{\sqrt{q}}}{[N]_q} = \pm \left(\frac{\tan(N\theta/2)}{\tan(\theta/2)} - 1 \right). \tag{3.14}$$

This case corresponds to the ‘string solutions’ of [13, 14]. The angle θ is restricted by the requirement of non-negativity of the elements A_n^2 (A_n^2) $\forall n \in (1, \dots, N-1)$:

$$A_n^2 = -A_n^2 = \frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)} \left(\frac{\tan(N\theta/2)}{\tan(n\theta/2)} - 1 \right). \tag{3.15}$$

For the regime $\mu < 0$, the positivity $A_n^2 > 0$ holds in the domain $-\pi/N < \theta < \pi/N$, whereas for the choice $\mu > 0$ the positivity $A_n^2 > 0$ is satisfied, for instance, in the restricted region $\pi/N < \theta < \pi/(N-1)$, $-\pi/(N-1) < \theta < -\pi/N$. This class of representation matrices is symmetric with respect to the minor diagonal: $A_{N-k} = A_k$, $A_{N-k} = A_k$.

(ii) The root of unity values of $q = \exp(i2\pi/N)$ restricts $p(>0)$ over a range. The solutions corresponding to this class are novel. We list some low-dimensional Hermitian representations for $\mu > 0$, as example,

$$a^\dagger = \begin{pmatrix} 0 & \sqrt{2-p} & 0 \\ & 0 & \sqrt{p} \\ & & 0 \end{pmatrix}, \quad a^\dagger = \begin{pmatrix} 0 & \sqrt{2-p} & 0 & 0 \\ & 0 & \sqrt{2} & 0 \\ & & 0 & \sqrt{p} \\ & & & 0 \end{pmatrix},$$

$$a^\dagger = \begin{pmatrix} 0 & \sqrt{2-p} & 0 & 0 & 0 \\ & 0 & \sqrt{2(1+(2-p)\sin\frac{\pi}{10})} & 0 & 0 \\ & & 0 & \sqrt{2(1+p\sin\frac{\pi}{10})} & 0 \\ & & & 0 & \sqrt{p} \\ & & & & 0 \end{pmatrix}. \tag{3.16}$$

The elements $A_n^2 \forall n \in (1, \dots, N-1)$ for representations (3.16) read

$$A_n^2 = \frac{\sin^2(n\pi/N)}{\sin^2(\pi/N)} (1 + (p-1)f_n), \quad f_n = \frac{\tan(\pi/N)}{\tan(n\pi/N)}. \tag{3.17}$$

The sequence $\{f_n|n = 1, \dots, N - 1\}$ is bounded as $-1 \leq f_n \leq 1$. The Hermiticity requirement now restricts the order parameter as $0 < p < 2$. The value of p being positive definite, the Hermitian representations of this class do not exist for $\mu < 0$.

(2) *Cyclic representations*

The cyclic finite-dimensional representations may be ensured by identifying the states $|N\rangle$ and $|0\rangle$. In particular, this makes the eigenvalues of the operators aa^\dagger and $a^\dagger a$ on the states $|N\rangle$ and $|0\rangle$ same. For the $\mu > 0$ case, we obtain

$$A_{N+1}^2 = p, \quad A_N^2 = r. \tag{3.18}$$

Eliminating r from the relations in (3.18), we obtain

$$[N]_{\sqrt{q}}^2 p = \frac{2}{2 - [2]_q} [N]_{\sqrt{q}}^2.$$

For the choice $[N]_{\sqrt{q}} \neq 0$, it follows $p = 2(2 - [2]_q)^{-1}$. It then turns out that $r = p$, and $A_n = \sqrt{p} \forall n$. This leads to $Z|n\rangle = 0$ [13]. This case is uninteresting. Alternate choice $[N]_{\sqrt{q}} = 0$, i.e. $q^N = 1$ yields a class of two-parametric (p, r) N -dimensional representations

$$a^\dagger = \begin{pmatrix} 0 & \sqrt{2-p-r} & 0 \\ \sqrt{r} & 0 & \sqrt{p} \\ 0 & 0 & 0 \end{pmatrix}, \quad a^\dagger = \begin{pmatrix} 0 & \sqrt{2-p} & 0 & 0 \\ 0 & 0 & \sqrt{2-r} & 0 \\ \sqrt{r} & 0 & 0 & \sqrt{p} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$a^\dagger = \begin{pmatrix} 0 & \sqrt{2(1+r \sin \frac{\pi}{10}) - p} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2(1+(2-p-r) \sin \frac{\pi}{10})} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2(1+p \sin \frac{\pi}{10}) - r} & 0 \\ \sqrt{r} & 0 & 0 & 0 & \sqrt{p} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.19}$$

These representations correspond to the ‘loop solutions’ in [13, 14] and possess the symmetry

$$A_n(p, r) = A_{N+1-n}(r, p). \tag{3.20}$$

To examine their non-negativity the elements $\{A_n^2|n \in (1, \dots, N)\}$ may be recast by isolating their symmetric part $\mathcal{S}_n(p+r)$ as follows:

$$A_n^2 = \mathcal{P}_n p + \mathcal{S}_n(p+r), \quad \mathcal{P}_n = [n]_q + [n-1]_q,$$

$$\mathcal{S}_n(\mathcal{X}) = -[n-1]_q \mathcal{X} + 2 \left[\frac{n-1}{2} \right]_q [n]_{\sqrt{q}}. \tag{3.21}$$

Due to the symmetry (3.20) we only need to check the non-negativity of the elements A_n^2 for $n = 1, \dots, N/2((N+1)/2)$ for an even (odd) N . In this domain we have $\mathcal{P}_n \geq 0$, and

$$\mathcal{S}_n(p+r) = \frac{\sin^2((n-1)\pi/N)}{\sin^2(\pi/N)} (1 + (1-p-r)f_{n-1}), \tag{3.22}$$

where the sequence $\{f_n\}$ has been defined in (3.17). In the present context the entries of the sequence $\{f_{n-1}\}$, where $n = 2, \dots, N/2((N+1)/2)$ for an even (odd) N , are bounded as $0 < f_{n-1} \leq 1$. This yields the parametric values $0 < p, 0 < r, p+r < 2$ for the required non-negativity. For the odd- N case, the symmetric element reads

$$A_{\frac{N+1}{2}}^2 = \frac{1}{2 \cos(\pi/N)} \left(\frac{1}{2 \sin^2(\pi/2N)} - p - r \right). \tag{3.23}$$

It may be noted that the solutions obtained in (3.16) may be obtained as the $r \rightarrow 0$ limiting case of the present cyclic representations. The corresponding matrix structures are, however, of different ranks, and in that sense inequivalent. Moreover, the cyclic property of solutions (3.19) is *lost* when $r = 0$ is substituted in them. Hermitian cyclic representations do not exist for the $\mu < 0$ case as the elements $\{A_n^2 | n = 1, \dots, N\}$ are not non-negative. For instance, we have

$$A_{\frac{N}{2}}^2 \Big|_{\text{even } N} = -r - \frac{1}{\sin^2(\pi/N)}, \quad A_{\frac{N+1}{2}}^2 \Big|_{\text{odd } N} = -\frac{1}{2 \cos(\pi/N)} \left(\frac{1}{2 \sin^2(\pi/2N)} + p + r \right). \tag{3.24}$$

(3) *Infinite-dimensional representations*

Turning toward the infinite-dimensional representations we discuss the special cases where we note the existence of such Hermitian representations. The identity

$$2 \left[\frac{n-1}{2} \right]_q [n]_{\sqrt{q}} = [n]_{\sqrt{q}}^2 - [n]_q$$

allows us to rewrite the elements A_n^2 for the Fock states ($r = 0$) in the $\mu > 0$ case as

$$A_n^2 = [n]_{\sqrt{q}}^2 + (p-1)[n]_q. \tag{3.25}$$

For the choice $p = 1$, therefore, the Hermiticity of the infinite set of basis states with $n = 1, 2, \dots, \infty$ is guaranteed. To establish a range of values of the order parameter p around $p = 1$ that preserves the positivity $A_n^2 > 0$, we use (3.13) to obtain

$$A_n^2 = \frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)} (1 + (p-1)f_n), \quad f_n = \frac{\tan(\theta/2)}{\tan(n\theta/2)}. \tag{3.26}$$

The entries of the sequence $\{f_n | n = 1, 2, \dots, \infty\}$ for any positive *integral* value of n remain *finite* for the choice $\theta = 2\pi/\mathcal{N}$, where \mathcal{N} is an *irrational* number. Employing the finite maximum and minimum entries of the said sequence a range of values of p may now be easily established for which Fock-state representations are Hermitian and well defined.

Another possible scenario is to turn to the quasi-classical states mentioned in the context of (2.13). Using the classical limit $q \rightarrow 1$ in the context of the (p, r) two-parametric states for the regime $\mu > 0$ we obtain

$$A_n^2 = n^2 + (p-r-1)n + r \quad \forall n = 1, 2, \dots, \infty. \tag{3.27}$$

For the choice $p \geq r+1$, these two-parametric states maintain $A_n^2 > 0$ ensuring the Hermiticity of the representations. These infinite-dimensional quasi-classical states reflect, in some sense, noncommutative spaces bordering on the commutative regime. Hermitian infinite-dimensional representations for the $\mu < 0$ case do not exist.

We have studied so far the case of $\mu \neq 0$. To complete the study of Hermitian representations, we next investigate the case of $\mu = 0$. Although the algebra of this case does not correspond to a deformed parafermion, one can repeat the analysis used earlier. We use the trilinear relation

$$W(W^\dagger)^2 - [2]_q W^\dagger W W^\dagger + (W^\dagger)^2 W = 0, \tag{3.28}$$

and assume a state $|0\rangle$ subject to the conditions

$$W W^\dagger |0\rangle = p|0\rangle, \quad W^\dagger W |0\rangle = r|0\rangle, \quad \langle 0|0\rangle = 1. \tag{3.29}$$

Using the same arguments applied earlier the positivity $p, r > 0$ follows, and the states are given by

$$W|n\rangle = ([n]_q p - [n-1]_q r)^{1/2} |n-1\rangle \equiv w_n |n-1\rangle, \quad (3.30)$$

$$W^\dagger|n\rangle = w_{n+1} |n+1\rangle. \quad (3.31)$$

This leads to

$$W^\dagger W|n\rangle = w_n^2 |n\rangle, \quad W W^\dagger|n\rangle = w_{n+1}^2 |n\rangle. \quad (3.32)$$

The eigenvalue of the Casimir operator reads

$$C_F|n\rangle = \frac{p^2 + r^2 - [2]_q p r}{2 - [2]_q} |n\rangle. \quad (3.33)$$

Relations parallel to (3.30)–(3.33) also hold for the state $| -n \rangle$.

Fock-type representation is obtained by requiring $W|0\rangle = 0$, i.e. $r = 0$. Implementing $w_N = 0$ we may truncate the Fock-type representation and obtain a finite N -dimensional one. There is a slight difference from the case of $\mu \neq 0$. The equation

$$w_N = \sqrt{[N]_q p} = 0, \quad p > 0$$

may be solved only for the choice $[N]_q = 0$. Namely, it is possible to obtain finite-dimensional Hermitian representations from the Fock-type one for $q^{2N} = 1$, i.e. $q = \exp(i\pi/N)$. The expression for w_n^2 is found to be non-negative:

$$w_n^2 = \frac{\sin(n\pi/N)}{\sin(\pi/N)} p > 0 \quad \forall n \in (1, \dots, N-1). \quad (3.34)$$

Such a truncation for a generic q does not exist.

Now we investigate the possible realizations of the cyclic representations for the $\mu = 0$ case. Conditions for cyclic representation are given by

$$w_N^2 = r, \quad w_{N+1}^2 = p. \quad (3.35)$$

Above equations may be recast as

$$[N]_q p = ([N-1]_q + 1)r, \quad ([N+1]_q - 1)p = [N]_q r. \quad (3.36)$$

Eliminating r from (3.36) we obtain a relation for p :

$$q^{-N}(1 - q^N)^2 p = 0.$$

Therefore the cyclic representations may exist only if $q^N = 1$. By computation, however, it may be seen that negative values of w_n^2 exist:

$$w_{\frac{N}{2}}^2 \Big|_{\text{even } N} = -r, \quad w_{\frac{N+1}{2}}^2 \Big|_{\text{odd } N} = -\frac{p+r}{2 \cos(\pi/N)}. \quad (3.37)$$

This eliminates the possibility of having Hermitian cyclic representations for the $\mu = 0$ case.

Lastly, paralleling the $\mu > 0$ case the infinite-dimensional representations for the choice $\mu = 0$ exist for the said semi-classical states in the $q \rightarrow 1$ limit as the elements

$$w_n^2 \Big|_{q \rightarrow 1} = n(p-r) + r \quad \forall n = 1, 2, \dots, \infty \quad (3.38)$$

are non-negative for $p \geq r$.

4. Concluding remarks

We introduced a viewpoint that regards fuzzy torus algebra as a q -deformed parafermion. Based on this picture, we investigated Hermitian representations of the fuzzy torus under the assumptions (3.1) and normalizability of the representation basis. All known representations in [13, 14] were recovered and some new representations (both finite and infinite dimensional) were discovered. To summarize, the finite-dimensional Hermitian representations of algebra (2.10) and (2.12) may be classified as follows. (i) For a generic value of q , the representations exist with N -dependent-order parameter p for both the $\mu < 0$ and $\mu > 0$ regimes. (ii) For the root of unity values of $q = \exp(i2\pi/N)$, the Hermitian Fock-type representations given in (3.16) exist only for the choice $\mu > 0$. (iii) Two parametric cyclic representations given in (3.19) exist for root of unity values of $q = \exp(i2\pi/N)$ for the $\mu > 0$ case. In the $r \rightarrow 0$ limit these representations reduce to the Fock-type solutions (3.16). Nevertheless, the matrix structures of the two classes of solutions are of different ranks, and the cyclic property of solutions (3.19) is lost on substituting $r = 0$ in them. As these two categories of solutions reflect the allowed noncommutative structure of spaces, their geometric properties are likely to be quite distinct.

Turning toward the existence of an infinite-dimensional Hermitian Fock-type representation discussed here, we note that these solutions allow us to introduce coherent states of the fuzzy torus as eigenstates of the operator a . The coherent state will lead to developing a star-product on the fuzzy torus by the method of [23]. This will be discussed in a separate publication.

We have seen that the fuzzy torus algebra admits interpretations as a q -deformed Lie algebra [14], or a q -deformed parafermion. These interpretations may not be the only possibilities. One can derive the following set of relations from (2.2):

$$\begin{aligned} [X, Y] &= i\hbar Z, & [Z, \varphi] &= 0, \\ [Z, X] &= i\hbar\{Y, \varphi\}, & \{Z, X\} &= i\hbar^{-1}[Y, \varphi], \\ [Z, Y] &= -i\hbar\{X, \varphi\}, & \{Z, Y\} &= -i\hbar^{-1}[X, \varphi]. \end{aligned} \quad (4.1)$$

This set looks like a variant of the Sklyanin algebra [24]. The implications of this viewpoint are not clear at present. It is also important to develop a supersymmetric extension of the fuzzy torus, and fuzzy surfaces of higher genus. Their relations with generalizations of parafermions are of interest.

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